Why is scurl($\mathbb{F}$) "circulation per area" at (x,y)?

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Math 53]

I will explain this (very sketchily!) in the case of a tiny square without using or proving Green's Theorem, since I want to use this intuition to justify Green's Theorem.

Say $\mathbb{F} = \langle P, Q \rangle : \mathbb{R}^2 \to \vec{\mathbb{R}}^2$ is a vector field on $\mathbb{R}^2$. Let D be a tiny ε-by-ε square at a point (x,y) (see figure). First, we need two approximations*:

$$Q(x+\varepsilon, y) - Q(x,y) \overset{\varepsilon\to 0}{\sim} \left\langle \begin{matrix} Q_x(x,y) \\ Q_y(x,y) \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} (x+\varepsilon)-x \\ y-y \end{matrix} \right\rangle,$$

therefore, omitting "(x,y)" where appropriate,

$$\boxed{Q(x+\varepsilon, y) \overset{\varepsilon\to 0}{\sim} Q + \varepsilon Q_x},$$

and similarly

$$\boxed{P(x, y+\varepsilon) \overset{\varepsilon\to 0}{\sim} P + \varepsilon P_y}$$

Write ∂D as $\vec{C_1} + \vec{C_2} - \vec{C_3} - \vec{C_4}$, as shown.

* Using the Linear Approximation Formula for gradients. I have taken this approach because I think it is the most physically intuitive, and illustrates the power of approximation when used correctly.

Now, let's approximate $\int_{\partial D} \mathbb{F}\cdot$ using a Riemann Sum with only one term for each $\underset{\rightharpoondown}{C_i}$:

$$\int_{\underset{\rightharpoondown}{C_1}} P\,dx + Q\,dy \overset{\varepsilon\to 0}{\sim} P(x,y)\,dx\langle \varepsilon, 0\rangle + Q(x,y)\,dy\langle \varepsilon, 0\rangle = \boxed{\varepsilon P}$$

$$\int_{\underset{\rightharpoondown}{C_2}} P\,dx + Q\,dy \overset{\varepsilon\to 0}{\sim} P(x+\varepsilon,y)\,dx\langle 0, \varepsilon\rangle + Q(x+\varepsilon,y)\,dy\langle 0, \varepsilon\rangle = \varepsilon Q(x+\varepsilon,y) \overset{\varepsilon\to 0}{\sim} \boxed{\varepsilon(Q+\varepsilon Q_x)}$$

$$\int_{\underset{\rightharpoondown}{C_3}} P\,dx + Q\,dy \overset{\varepsilon\to 0}{\sim} P(x,y+\varepsilon)\,dx\langle \varepsilon, 0\rangle + Q(x,y+\varepsilon)\,dy\langle \varepsilon, 0\rangle = \varepsilon P(x,y+\varepsilon) \overset{\varepsilon\to 0}{\sim} \boxed{\varepsilon(P+\varepsilon P_y)}$$

$$\int_{\underset{\rightharpoondown}{C_4}} P\,dx + Q\,dy \overset{\varepsilon\to 0}{\sim} P(x,y)\,dx\langle 0, \varepsilon\rangle + Q(x,y)\,dy\langle 0, \varepsilon\rangle = \boxed{\varepsilon Q}$$

$$\therefore \operatorname{Circ}_{\mathbb{F}}(D) := \int_{\partial D} P\,dx + Q\,dy \overset{\varepsilon\to 0}{\sim} \varepsilon P + \varepsilon(Q+\varepsilon Q_x) - \varepsilon(P+\varepsilon P_y) - \varepsilon Q$$

$$= \varepsilon^2 (Q_x - P_y) = \boxed{\operatorname{area}(D)\cdot \operatorname{Scurl}(\mathbb{F})(x,y)},$$

and in fact $\boxed{\operatorname{Scurl}(\mathbb{F})(x,y) = \lim_{\varepsilon\to 0} \frac{\operatorname{Circ}_{\mathbb{F}}(D)}{\operatorname{area}(D)}}$.