Why is \( \text{curl}(\mathbf{F}) \) "circulation per area" at \((x,y)\)?

I will explain this (very sketchily!) in the case of a tiny square
without using or proving Green's Theorem, since I want to use
this intuition to justify Green's Theorem.

Say \( \mathbf{F} = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2 \) is a vector field on \( \mathbb{R}^2 \). Let
\( D \) be a tiny \( \varepsilon \)-by-\( \varepsilon \) square at a point \((x,y)\)
(see figure). First, we need two approximations:

\[
Q(x+\varepsilon, y) - Q(x,y) \sim \frac{\varepsilon \partial Q}{\partial x}(x,y) \cdot (x+\varepsilon - x) \quad (x,y) \sim Q + \varepsilon \partial Q \quad \text{and similarly}
\]

\[
P(x,y+\varepsilon) \sim P + \varepsilon \partial P_y
\]

Write \( \partial D \) as \( C_1 + C_2 - C_3 - C_4 \), as shown.

* Using the Linear Approximation Formula for gradients.
I have taken this approach because I think it is the
most physically intuitive, and illustrates the power of
approximation when used correctly.
Now, let's approximate $\int_{\partial D} \nabla \times \mathbf{F} \cdot d\mathbf{S}$ using a Riemann sum with only one term for each $\xi_i$:

$$\int_{\partial D} \nabla \times \mathbf{F} \cdot d\mathbf{S} \approx \sum_{\xi_i} \mathbf{F}(\mathbf{x}_i) \cdot d\mathbf{S}_i = \mathbf{F}(\mathbf{x}_i) \cdot d\mathbf{S}_i$$

For each $\xi_i$, we have:

$$\int_{\partial D} \nabla \times \mathbf{F} \cdot d\mathbf{S} \approx \sum_{\xi_i} \mathbf{F}(\mathbf{x}_i) \cdot d\mathbf{S}_i = \mathbf{F}(\mathbf{x}_i) \cdot d\mathbf{S}_i$$

Thus, for each $\xi_i$, we can approximate the circulation as:

$$\int_{\partial D} \nabla \times \mathbf{F} \cdot d\mathbf{S} \approx \sum_{\xi_i} \mathbf{F}(\mathbf{x}_i) \cdot d\mathbf{S}_i = \mathbf{F}(\mathbf{x}_i) \cdot d\mathbf{S}_i$$

And in fact:

$$\text{Scurl} \mathbf{F}(\mathbf{x}, \mathbf{y}) = \lim_{\varepsilon \to 0} \frac{\text{Circ}_{\varepsilon}(\mathbf{D})}{\text{Area}(\mathbf{D})}$$

For a closed path $\mathbf{C}$:

$$\text{Circ}(\mathbf{C}) = \oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{S}$$