Why is Circulation Additive?

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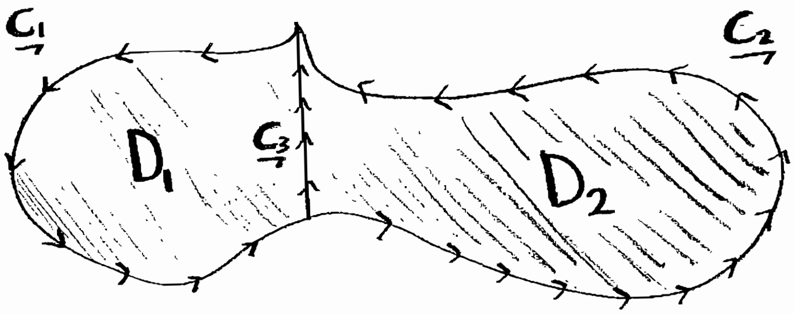
Math 53]



Say $D \subseteq \mathbb{R}^2$ is divided into regions D_1, D_2 as shown, and $\mathbb{F}$ is a vector field on $\mathbb{R}^2$ (not drawn).

Recall that the "circulation of $\mathbb{F}$ around D" is defined as

$\operatorname{circ}_{\mathbb{F}}(D) = \int_{\partial D} \mathbb{F}\cdot$, an oriented curve integral.

<u>Fact:</u> $\operatorname{circ}_{\mathbb{F}}(D_1) + \operatorname{circ}_{\mathbb{F}}(D_2) = \operatorname{circ}_{\mathbb{F}}(D)$. Why?

$\partial D_1 = \underrightarrow{C_1} + \underrightarrow{C_3}$, $\underrightarrow{\partial D_2} = \underrightarrow{C_2} - \underrightarrow{C_3}$, and $\partial D = \underrightarrow{C_1} + \underrightarrow{C_2}$

(these are "formal" sums, which just indicate that we'll be adding integrals), i.e.

$$\int_{\partial D_1} \mathbb{F}\cdot = \int_{\underrightarrow{C_1}} \mathbb{F}\cdot + \int_{\underrightarrow{C_3}} \mathbb{F}\cdot \,, \quad \int_{\partial D_2} \mathbb{F}\cdot = \int_{\underrightarrow{C_2}} \mathbb{F}\cdot - \int_{\underrightarrow{C_3}} \mathbb{F}\cdot \,, \text{ and}$$

adding gives $\int_{\underrightarrow{C_1}} \mathbb{F}\cdot + \int_{\underrightarrow{C_2}} \mathbb{F}\cdot = \int_{\partial D} \mathbb{F}\cdot$

Intuitively, the "help" we might get from $\mathbb{F}$ as we move a particle along $\underset{\rightarrow}{C_3}$ is cancelled by the "hurt" as we move the particle along $-\underset{\rightarrow}{C_3}$.

I.e., the work done by $\mathbb{F}$ on a particle which moves around D is the same as the work done by $\mathbb{F}$ on a particle which moves around D_1 and then around D_2:

This additivity is what allows us to think of Green's Theorem as "adding up circulation around the points of D":

$$\boxed{\iint_D \underbrace{\text{scurl}(\mathbb{F})}_{\substack{\text{circulation per}\\ \text{area near}\\ \text{each point}}} \underbrace{dA}_{\substack{\text{tiny}\\ \text{area}\\ \text{around}\\ \text{each}\\ \text{point}}} = \underbrace{\int_{\partial D} \mathbb{F}\cdot}_{\substack{\text{total circulation}\\ \text{around D}}}}$$