Surface integrals of scalar fields wrt surface area.

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Math 53

Recall that $d\text{¢}$ of a (usually tiny) vector $\vec{a}$ is just the (positive) length of $\vec{a}$, e.g. $d\text{¢}\left\langle \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \right\rangle = \sqrt{14}$. Similarly, we define $d\$$ of an ordered parallogram $\vec{a}, \vec{b}$

area of $\vec{a}, \vec{b}$, i.e. $d\$(\vec{a}, \vec{b}) = \boxed{\|\vec{a} \times \vec{b}\|}$

Recall that $\int_C f\, d\text{¢}$ is defined by a (Riemann) sum, where f takes in points $\underset{\bullet}{x}_i$ and $d\text{¢}$ takes in vectors $\vec{c}_i$ (tiny bits of curve; direction doesn't affect $d\text{¢}(\vec{c}_i) = \|\vec{c}_i\|$!), and we add up the terms $f(\underset{\bullet}{x}_i)\, d\text{¢}(\vec{c}_i)$.

Similarly, $\iint_S f\, d\$$ is defined by a (Riemann) sum, where f takes in points $\underset{\bullet}{x}_i$, and $d\$$ takes in parallelograms $\vec{a}, \vec{b}$ (tiny bits of surface; order/direction doesn't affect $d\$(\vec{a}, \vec{b}) = \|\vec{a} \times \vec{b}\|$!), and we add up the terms $f(\underset{\bullet}{x}_i)\, d\$(\vec{a}, \vec{b})$.

You can just think of $\iint_S f\,d\$$ like a "sum of outputs of f from inputs on S," which is good for most scientific applications, but if you insist on thinking geometrically, it is the (signed) volume between S and $\text{graph}(f) \subseteq \mathbb{R}^3 \times \mathbb{R}^1 = \mathbb{R}^4$.

To compute, we usually parametrize S once, say $\mathbf{r} : D \xrightarrow{1} S$ with D in $\mathbb{R}^2_{u,v}$ and S in $\mathbb{R}^3_{x,y,z}$)

and think of S being made up of the tiny parallelograms "$\mathbf{r}_u du, \mathbf{r}_v dv$" (from the LAF!).

Since $d\$(\mathbf{r}_u du, \mathbf{r}_v dv) = \|\mathbf{r}_u du \times \mathbf{r}_v dv\| = \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv$,

we get the formula:

short for $f(x(u,v), y(u,v), z(u,v))$

$$\boxed{\iint_S f\,d\$ = \iint_D f(u,v)\,\|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv}$$

(iii) Please see Stewart 16.7 for examples!

Remarks: 1) The (Riemann) sum definition of $\iint_S f \, dS$ does not involve parametrization! So the answer doesn't depend on any parametrization of S.

2) Recall that $d\ell$ had a "stand alone" definition $d\ell = \sqrt{(dx)^2 + (dy)^2}$. So does dS, but it involves a "wedge product" operation:

$$dS = \sqrt{(dx \wedge dy)^2 + (dy \wedge dz)^2 + (dz \wedge dx)^2},$$

which is easy to understand (with the right explanation!), but it's not part of this course.