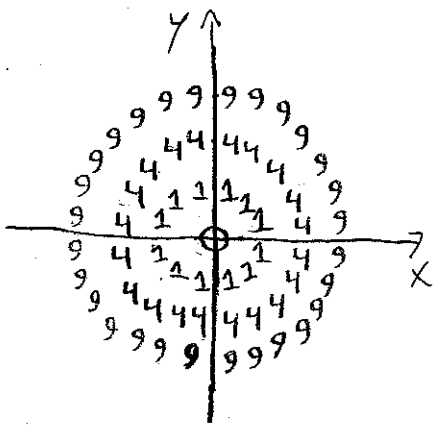
Scalar fields, vector fields, and differential One-forms (covector fields). [Andrew Critch, Math 53]

A scalar field on $\mathbb{R}^3$ is just a function $f : \underset{\bullet}{\mathbb{R}^3} \to \mathbb{R}^1$ (usually differentiable), i.e. f has point inputs and scalar outputs. Example on $\underset{\bullet}{\mathbb{R}^2}$: $f(x,y) = \boxed{x^2+y^2}$



You can think visually about a scalar field by imagining each scalar output sitting at its input (so you don't have to think of graphs in $\underset{\bullet}{\mathbb{R}^4}$!)

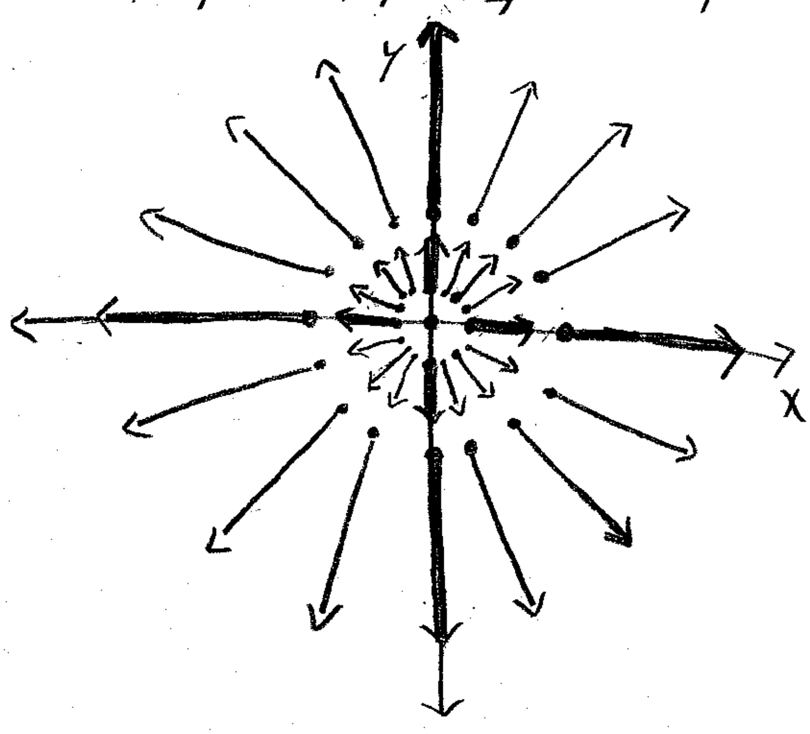
A vector field on $\mathbb{R}^3$ is just a function $F : \underset{\bullet}{\mathbb{R}^3} \to \underrightarrow{\mathbb{R}^3}$, (usually differentiable), i.e. F has $\boxed{\text{point}}$ inputs and $\boxed{\text{vector}}$ outputs.

The $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ components of these vector outputs will be scalars (functions) that depend on the point inputs, so we can write any vector field on $\mathbb{R}^3$ in the form $F(\underset{\bullet}{x}) = \langle P(\underset{\bullet}{x}), Q(\underset{\bullet}{x}), R(\underset{\bullet}{x})\rangle$, or simply

$$\boxed{F = \langle P, Q, R\rangle = P\underset{\sim}{i} + Q\underset{\sim}{j} + R\underset{\sim}{k}}.$$

Example: The gradient ∇f of any scalar field f is a vector field! After all, $\nabla f(\underset{\bullet}{x}_0)$ is a vector for each point input $\underset{\bullet}{x}_0$. You can visualize a vector field by thinking of each vector output sitting at its point input. E.g. on $\mathbb{R}^2$, say

$F(x,y) = \langle 2x, 2y\rangle = 2x\underset{\sim}{i} + 2y\underset{\sim}{j} = \nabla(x^2+y^2)$:



Most arbitrary vector fields do <u>not</u> arise as gradients of scalar fields; e.g. $F = y\underset{\rightarrow}{i} - x\underset{\rightarrow}{j}$:

<u>if</u> we had $\nabla f = F$, i.e. $f_x = \boxed{y}$ and $f_y = \boxed{-x}$,

then $f_{xy} = \boxed{1} \neq \boxed{-1} = f_{yx}$, contradicting

$\boxed{\text{Clairaut's Theorem}}$. Thus $\langle y, -x \rangle$ is not the gradient of any $\boxed{\text{scalar field}}$ (more on gradients later).

A <u>covector field</u> or <u>(differential) one-form</u> is just the "dot" or "dual" of a vector field,

$$F\cdot = \langle P, Q, R\rangle \cdot = (P\underset{\rightarrow}{i} + Q\underset{\rightarrow}{j} + R\underset{\rightarrow}{k})\cdot = P\,dx + Q\,dy + R\,dz.$$

It is <u>very important</u> to understand that, although the covectors dx, dy, dz take $\boxed{\text{vectors}}$ as inputs, the $\boxed{\text{scalar}}$ functions P, Q, R take $\boxed{\text{points}}$ as inputs.

For example, consider the one-form (covector field)

$$F\bullet = \langle x^2, xy, yz\rangle\bullet = x^2dx + xydy + yzdz.$$

Then $(x^2dx + xydy + yzdz)^{\text{of}}\begin{pmatrix}2\\3\\4\end{pmatrix} = \boxed{4dx + 6dy + 12dz}$,

Whereas $(x^2dx + xydy + yzdz)^{\text{of}}\left\langle\begin{matrix}4\\5\\6\end{matrix}\right\rangle = \boxed{4x^2 + 5xy + 6yz}$.

It's like the coordinate functions x, y, z extract the components of $\boxed{\text{points}}$, whereas the covectors dx, dy, dz extract the components of $\boxed{\text{vectors}}$. More examples:

If $F\bullet = \langle y, -x\rangle\bullet$, then $(F\bullet)^{\text{of}}\begin{pmatrix}3\\4\end{pmatrix} = \boxed{\langle 4, -3\rangle}$, and $(F\bullet)^{\text{of}}\left\langle\begin{matrix}5\\6\end{matrix}\right\rangle = \boxed{5y - 6x}$. Evaluate $\sin(x)dx - \cos(y)dy$

at the point $\begin{pmatrix}\pi\\\pi\end{pmatrix}$: $\boxed{dy}$

on the vector $\left\langle\begin{matrix}\pi\\\pi\end{matrix}\right\rangle$: $\boxed{\pi\sin(x) - \pi\cos(y)}$

It is not usually important in vector calculus exercises to actually evaluate one-forms at particular points / vectors, but it helps a great deal with understanding them conceptually.