

How do the $\nabla \times$ and $\nabla \cdot$ notations work?

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The symbol ∇ used for the gradient operator is called “nabla”. I am not a fan of the nabla notations $\nabla \times$ and $\nabla \cdot$ for curl and div, but I will explain them anyway :) These symbols convey little or no physical intuition, but a lot of people use them!

1.1 The “del” operators

The first idea, which I do like, is to think of the operators

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y}, \quad \text{and} \quad \partial_z = \frac{\partial}{\partial z}$$

(called “del sub x”, “del sub y”, and “del sub z”) as objects on their own, which differentiate *scalar fields* appearing at their right. Do not confuse them with dx and dy , which have straight d 's and no subscript!

For example, if P and Q are scalar fields, then we could write

$$\begin{aligned}(\partial_x + 2\partial_y)(P + Q) &= \partial_x(P) + 2\partial_y(P) + \partial_x(Q) + 2\partial_y(Q) \\ &= P_x + 2P_y + Q_x + 2Q_y\end{aligned}$$

We can also write $\partial_x \partial_x = \partial_{xx}$, $\partial_x \partial_y = \partial_{xy}$, etc., so for example

$$\begin{aligned}\partial_x(\partial_x + 2\partial_y)P &= \partial_{xx}(P) + 2\partial_{xy}(P) \\ &= P_{xx} + 2P_{xy}\end{aligned}$$

1.2 Writing ∇ as a “vector”

The next idea is to think of “nabla” like a “vector” (not physically, just notationally) with “operator components”:

$$\nabla = \left\langle \begin{matrix} \partial_x \\ \partial_y \\ \partial_z \end{matrix} \right\rangle, \quad \text{so} \quad \nabla f = \left\langle \begin{matrix} \partial_x \\ \partial_y \\ \partial_z \end{matrix} \right\rangle f = \left\langle \begin{matrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{matrix} \right\rangle = \left\langle \begin{matrix} f_x \\ f_y \\ f_z \end{matrix} \right\rangle$$

1.3 Nabla notation for curl and div.

If $\mathbf{F} = \langle P, Q, R \rangle$, we can write curl and divergence as follows:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \begin{matrix} \partial_x \\ \partial_y \\ \partial_z \end{matrix} \right\rangle \times \left\langle \begin{matrix} P \\ Q \\ R \end{matrix} \right\rangle = \left\langle \begin{matrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{matrix} \right\rangle$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \begin{matrix} \partial_x \\ \partial_y \\ \partial_z \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} P \\ Q \\ R \end{matrix} \right\rangle = P_x + Q_y + R_z$$

1.4 Trick proofs that $\text{curl } \nabla f = 0$ and $\text{div curl } \mathbf{F} = 0$

In nabla notation, these theorems look like

$$\nabla \times (\nabla f) = 0 \quad \text{and} \quad \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

which can be proven purely algebraically in the same way as

$$\mathbf{a} \times (\mathbf{a} \cdot \mathbf{c}) = 0 \quad \text{and} \quad \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0.$$

1.5 Why is this stuff so confusing?

These notations and proofs are not very physically enlightening, because they rely on algebraic tricks built into the definition of curl and the cross product, which are nowadays just a way of avoiding a much more natural operation called the *wedge product*. So if you are unsatisfied with all this algebra, don't worry, there is a much more organized way of doing it!

Using the wedge product, mathematicians in the early 20th century discovered a concept called the *exterior derivative* which completely replaces grad, curl, and div, and is in fact notationally, algebraically, and conceptually more efficient. If understood correctly, it even provides better physical intuition, but unfortunately physicists haven't caught on to it yet. So despite being easy, introductory vector calculus courses still do not include it. Maybe, if there is time in class or office hours near the end of the course, I will :)

1.6 Optional: nabla notation for the Jacobian

If you don't understand these next two sections, don't worry!

Say $\mathbf{F} = \langle P, Q, R \rangle$ is a (C^2) vector field on \mathbb{R}^3 . Since \mathbf{F} is a map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, it has a Jacobian matrix $\mathbb{J}_{\mathbf{F}}$. Let "nabla transpose" be the row matrix $\nabla^T = [{}_x\partial \quad {}_y\partial \quad {}_z\partial]$, where ${}_x\partial$ is like a backwards ∂_x , which differentiates scalar fields to its left: $(P)_{{}_x\partial} = P_x$. Then we can write the following (and similarly on \mathbb{R}^n):

$$\mathbb{J}_{\mathbf{F}} = \mathbf{F} \nabla^T = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} [{}_x\partial \quad {}_y\partial \quad {}_z\partial] = \begin{bmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{bmatrix}$$

1.7 The curl-Jacobian trick: $(\text{curl } \mathbf{F}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbb{J}_{\mathbf{F}} \mathbf{a}) \cdot \mathbf{b} - (\mathbb{J}_{\mathbf{F}} \mathbf{b}) \cdot \mathbf{a}$

This equation is fundamental: it is relatively easy to show that that the RHS approximates the circulation of \mathbf{F} around a small ordered parallelogram \mathbf{a}, \mathbf{b} starting at a point (x, y, z) , and then we can infer that the LHS does as well — this is the essence of Stokes' Theorem! Reversing dot products and using nabla, this equation looks like

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\nabla \times \mathbf{F}) &= (\mathbf{F} \nabla^T \mathbf{a}) \cdot \mathbf{b} - (\mathbf{F} \nabla^T \mathbf{b}) \cdot \mathbf{a} \\ &= (\mathbf{a} \cdot \nabla)(\mathbf{b} \cdot \mathbf{F}) - (\mathbf{b} \cdot \nabla)(\mathbf{a} \cdot \mathbf{F}) \end{aligned}$$

which can be proven purely algebraically in the same way as

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}).$$