Implicit vs. Parametric Equations

[Andrew Critch, Math 53, 09Su]

The difference between implicit and parametric equations is how we use them. In short,

Implicit equations are used to define level sets of functions

parametric equations are used to define ranges of functions. (the inputs of these functions are called parameters).

Examples

1) The equation $x^2+y^2=1$ defines a circle as a level set:
   - We identify a function $f(x,y)=x^2+y^2$, $f:\mathbb{R}^2\to\mathbb{R}^1$.
   - We select the set of inputs $(x,y)\in\mathbb{R}^2$ which make the equation $f(x,y)=1$ true, i.e. the level set $\{f=1\}$.
   - This set is a circle in $\mathbb{R}^2$!

   Thus, we say "$x^2+y^2=1$" is an implicit equation of a circle in $\mathbb{R}^2$.

2) The equations $x=\cos(t)$, $y=\sin(t)$ define a circle as a range:
   - We identify a function $\mathbf{r}(t)=(\cos t,\sin t)$, $\mathbf{r}:\mathbb{R}^1\to\mathbb{R}^2$
   - We select the set of all outputs $(\cos t,\sin t)\in\mathbb{R}^2$ of this function, i.e. range($\mathbf{r}$).
   - The set range($\mathbf{r}$) is a circle in $\mathbb{R}^2$!

Thus, we say "$x = \cos t,\ y = \sin t$" are parametric equations for a circle in $\mathbb{R}^2$. We say the variable t is a parameter, meaning it is an input of the function $\mathbf{r}(t) = (\cos t, \sin t)$ whose range we are considering.

3) The equation $x^2 + y^2 = 1 - z^2$ is an implicit equation of a sphere in $\mathbb{R}^3$. Although it is not quite in the level set format "function = constant", we could rearrange it to $x^2 + y^2 + z^2 = 1$. Alternatively, we can simply consider the set of "inputs" $(x, y, z) \in \mathbb{R}^3$ which make the original equation true, without rearranging or thinking about level sets.

4) The (two!) equations "$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}$" are implicit equations of a line in $\mathbb{R}^3$: the set of "inputs" $(x, y, z) \in \mathbb{R}^3$ making the equation true is a line.

5) The vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} t$ (corresponding to the three scalar equations $x = 1+4t$, $y = 2+5t$, $z = 3+6t$) is a *parametric* equation for a line in $\mathbb{R}^3$:

the function $\mathbf{r}(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} t$ has $\text{range}(\mathbf{r})$ = a line.

Here, t is called a *parameter*, because it lives in the *domain* of the parametrization).

Given particular equations, how can you decide whether to use them implicitly or parametrically?

One way is to decide whether variables are *coordinates* or *parameters*. It is traditional to use *x, y, and z* as *coordinates* on $\mathbb{R}^3$, and to use letters like *s and t* as *parameters*. Then, parametric equations look like

> "*coordinates* = functions of *parameters*", e.g.
>
> "$x = 1+t^2$"
>
> "$y = 2+t^3$"

This format is a clue to consider the *range* of the function $\mathbf{r}(t) = \begin{bmatrix} 1+t^2 \\ 2+t^3 \end{bmatrix}$.

By contrast, a graph has an equation of the form

> "coordinate = function of coordinates", e.g.
> "$z = x^3y + y^2$"

This format is a clue to consider the graph of the function $f(x,y) = x^3y + y^2$ (or perhaps the level set $g(x,y,z) = z - x^3y - y^3 = 0$) : a surface in $\mathbb{R}^3$.

Reread the examples above with these rules of thumb in mind. Here are some more examples to check your understanding (not to memorize!)

① $\begin{Bmatrix} x+y+z=1 \\ x+2y+3z=4 \end{Bmatrix}$ ② $\vec{x} = \begin{bmatrix} \sin(t) \\ \cos(t) \\ t \end{bmatrix}$ ③ $x^2+y^3+z^4=5$

④ $\begin{bmatrix} x^2+y^2-1 \\ 2x+y-z \end{bmatrix} = \vec{0}$ ⑤ $2x+3y=4$ ⑥ $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \cdot \vec{x} = 6$

⑦ $\vec{x} = \vec{x}_0 + \vec{v} \cdot t$ ⑧ $y = x^2$ ⑨ $x = t^2, y = t^3$

⑩ $xy = 6$ in $\mathbb{R}^3$

(answers next page)

Answers:

(2) parametric equation(s) (of a curve) in $\mathbb{R}^3$

(6) Implicit equation (of a plane) in $\mathbb{R}^3$

(8) Graph or implicit equation (of a curve) in $\mathbb{R}^2$

(10) Implicit equation (of a surface) in $\mathbb{R}^3$

(1) Implicit equations (of a line) in $\mathbb{R}^3$

(5) Implicit equation (of a line) in $\mathbb{R}^2$

(7) parametric equation (of a line) in $\mathbb{R}^n$ (n not specified!)

(3) Implicit equation (of a surface) in $\mathbb{R}^3$

(9) parametric equations (of a curve) in $\mathbb{R}^2$.

(4) Implicit equation(s) (of a curve) in $\mathbb{R}^3$.