How do Scurl/curl approximate parallelogram circulation?
[Andrew Critch, Math 53]

I will explain this (very sketchily!) without using or proving Stokes' Theorem, since I want to use this intuition to justify Stokes' Theorem.

Suppose that $\mathbb{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field on $\mathbb{R}^n$. Let D be a tiny ordered parallelogram $\vec{a}, \vec{b}$ at a point $\mathbf{x} \in \mathbb{R}^n$ (See Figure, drawn in the plane spanned by $\vec{a}, \vec{b}$ at $\mathbf{x}$). First, we need two approximations using the Jacobian matrix $\mathbb{J}_{\mathbb{F}}$:

$$\mathbb{F}(\mathbf{x}+\vec{a}) - \mathbb{F}(\mathbf{x}) \overset{\vec{a}\to 0}{\sim} \mathbb{J}_{\mathbb{F}}(\mathbf{x})\langle(\mathbf{x}+\vec{a})-\mathbf{x}\rangle,$$

therefore, omitting the input "$(\mathbf{x})$" for convenient writing,

$$\boxed{\mathbb{F}(\mathbf{x}+\vec{a}) \overset{\vec{a}\to 0}{\sim} \mathbb{F} + \mathbb{J}_{\mathbb{F}}\vec{a}},$$

and similarly

$$\boxed{\mathbb{F}(\mathbf{x}+\vec{b}) \overset{\vec{b}\to 0}{\sim} \mathbb{F} + \mathbb{J}_{\mathbb{F}}\vec{b}}.$$

Now, write $\partial D = \vec{C_1} + \vec{C_2} - \vec{C_3} - \vec{C_4}$, as shown.

Now, let's approximate $\int_{\partial D} \mathbb{F}\cdot$ using a Riemann Sum with only one term for each $\vec{C}_i$:

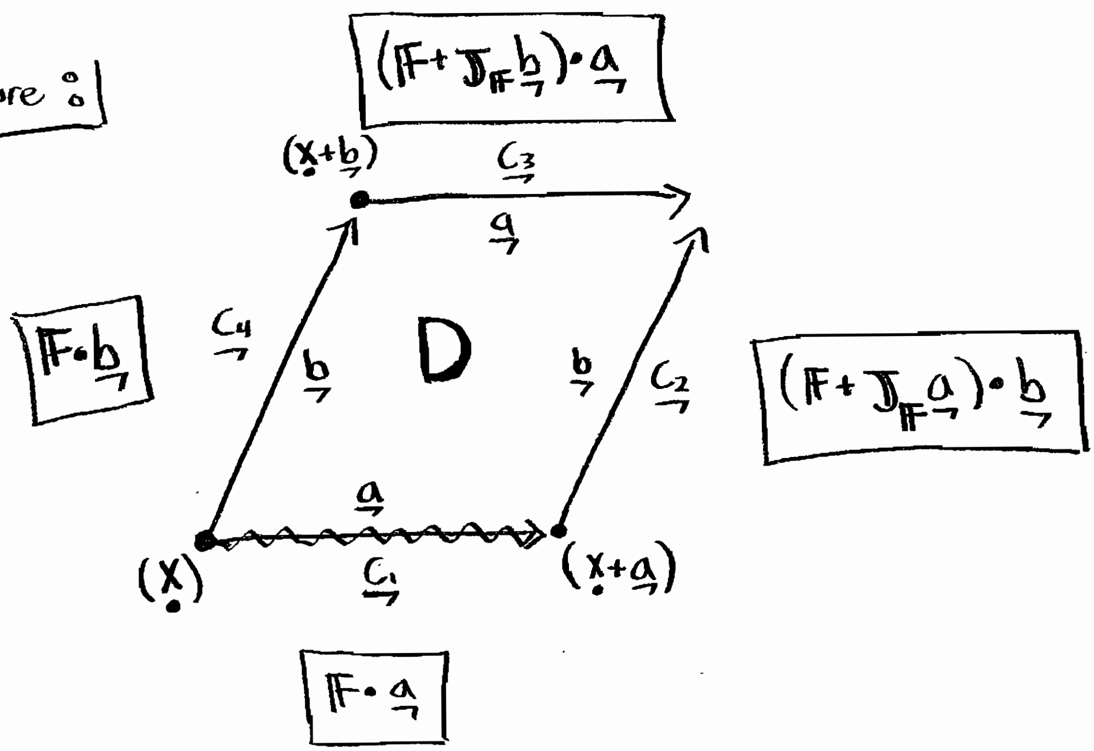
$$\int_{\vec{C}_1} \mathbb{F}\cdot \overset{\vec{a},\vec{b}\to 0}{\sim} \mathbb{F}(\underset{\bullet}{x})\cdot \vec{a} = \boxed{\mathbb{F}\cdot\vec{a}}$$

$$\int_{\vec{C}_2} \mathbb{F}\cdot \sim \mathbb{F}(\underset{\bullet}{x}+\vec{a})\cdot\vec{b} \sim \boxed{(\mathbb{F}+\mathbb{J}_{\mathbb{F}}\vec{a})\cdot\vec{b}}$$

$$\int_{\vec{C}_3} \mathbb{F}\cdot \sim \mathbb{F}(\underset{\bullet}{x}+\vec{b})\cdot\vec{a} \sim \boxed{(\mathbb{F}+\mathbb{J}_{\mathbb{F}}\vec{b})\cdot\vec{a}}$$

$$\int_{\vec{C}_4} \mathbb{F}\cdot \sim \mathbb{F}(\underset{\bullet}{x})\cdot\vec{b} = \boxed{\mathbb{F}\cdot\vec{b}}.$$

Figure :



Hence, the circulation of $\mathbb{F}$ around this parallelogram is

$$\int_{\partial D} \mathbb{F}\cdot = \int_{\underset{\rightarrow}{C_1}} \mathbb{F}\cdot + \int_{\underset{\rightarrow}{C_2}} \mathbb{F}\cdot - \int_{\underset{\rightarrow}{C_3}} \mathbb{F}\cdot - \int_{\underset{\rightarrow}{C_4}} \mathbb{F}$$

$$\overset{\underline{a},\underline{b}\to 0}{\sim} \mathbb{F}\cdot\underline{a} + (\mathbb{F} + \mathbb{J}_{\mathbb{F}}\underline{a})\cdot\underline{b} - (\mathbb{F} + \mathbb{J}_{\mathbb{F}}\underline{b})\cdot\underline{a} - \mathbb{F}\cdot\underline{b}$$

$$= \boxed{(\mathbb{J}_{\mathbb{F}}\underline{a})\cdot\underline{b} - (\mathbb{J}_{\mathbb{F}}\underline{b})\cdot\underline{a}}$$

Thus we have a formula for approximating parallelogram circulation <u>in $\mathbb{R}^n$.</u> In $\mathbb{R}^2$, writing $\mathbb{F} = \langle P, Q\rangle$, $\underline{a} = \langle a_1, a_2\rangle$, $\underline{b} = \langle b_1, b_2\rangle$, you can expand the RHS, and four terms cancel to leave

$$(Q_x - P_y)(a_1 b_2 - a_2 b_1) = \boxed{(\text{scurl}\,\mathbb{F})\det(\underline{a},\underline{b})}. \quad (\text{do it!})$$

In $\mathbb{R}^3$, 6 out of 18 terms on the RHS cancel, leaving

$$\boxed{(\text{curl}\,\mathbb{F})\cdot(\underline{a}\times\underline{b})}. \quad (\text{Tada!})$$

like 3 scurls   like 3 determinants

<u>Remark</u>: these cancellations have been organized together in the modern definition of "exterior derivative" (using wedge products "$\wedge$"), which generalizes grad, scurl, curl, and div.