Notes on Extrema, a (terribly) concise commentary on Stewart 14.7.

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Math 53, 09Su]

(Ⓜ indicates something you should read in the text.
☆ indicates my comments/summary)

Ⓜ $\boxed{\text{p. 923, def'n 1}}$ ☆ Terminology: if f "has a local min/max" "at (a,b)", then for clarity, I will call:

- The graph point $(a,b,f(a,b))$ a local maximum/minimum
- The input (a,b) a local maximizer/minimizer
- The output $f(a,b)$ a local max/min value

(The word "extremum" means "maximum or minimum". "Extrema", "maxima" and "minima" are the plural forms.)

Ⓜ $\boxed{\text{p. 923, def'n "critical point"}}$ ☆ I prefer the term "critical input" for clarity: an input where f_x and f_y are undef. or $= 0$

p. 923, Thm 2

easy to find!

$$\left\{\begin{matrix}\text{absolute}\\ \text{extremizers}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{local}\\ \text{extremizers}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{critical}\\ \text{inputs}\end{matrix}\right\}$$

☆ {absolute extremizers} ⊆ {local extremizers} ⊆ {critical inputs} (critical inputs " easy to find!)

intuitive idea: Say $(a,b)=(x_0,y_0)$ is a local maximizer.

- if $f_x(x_0,y_0)>0$, then increasing x_0
- if $f_x(x_0,y_0)<0$, then decreasing y_0
- if $f_y(x_0,y_0)>0$, then increasing y_0
- if $f_y(x_0,y_0)<0$, then decreasing y_0

increases f, so (x_0,y_0) is not a local maximizer.

hence f_x and f_y must be undef or $=0$ at (x_0,y_0).

p. 923, Exa. 1 / p. 924, Exa. 2

☆ Once we find a critical input (x_0,y_0), we want an easy way to check whether it is a local extremizer. For this, we need a common condition called "C^2", read "C-two", which stands for "continuous 2nd partials." It means that f_{xx}, f_{xy}, f_{yx}, and f_{yy} (exist and) are continuous.

Functions like polynomials, rational functions, logs, trig and exponential functions all have this property where they are defined.

$\boxed{\text{p. 924, "2}^{\text{nd}}\text{ der. test"}}$ ★ say $f = f(x,y)$, and: (only two variables!)

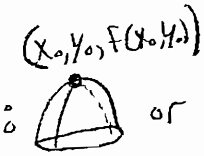
- f has "continous 2^{nd} partials" near (x_0, y_0)

- $\nabla f(x_0, y_0) = 0$

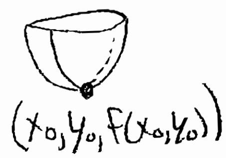
text denotes by "D"

- $H := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} (x_0, y_0)$, (called the "Hessian determinant") then:

★ $H > 0 \Rightarrow (x_0, y_0)$ is a "Dome input":



or



then:
- f_{xx} or $f_{yy} > 0 \Rightarrow$ convex up (local min)
- f_{xx} or $f_{yy} < 0 \Rightarrow$ convex down (local max)

★ $H < 0 \Rightarrow (x_0, y_0)$ is a "saddle input":
$\Rightarrow$ not a local extremizers

★ $H = 0 \Rightarrow$ dunno, use other methods to classify (x_0, y_0)

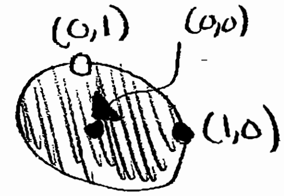
(★ Example of a local min with $|H| = 0$:
$f(x,y) = x^4 + y^4$ at $(0,0)$.)

$\boxed{\text{pp. 924-928, Exa. 3-6}}$

Absolute Extrema

$\boxed{\text{p. 928, def'n of Closed set and boundary point, with example.}}$

★ Another example: say $D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x^2+y^2 \le 1 \\ \text{and} \\ y<1 \end{matrix} \right\}$.



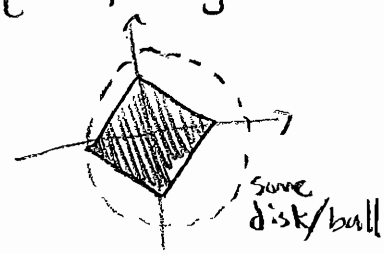
Then: $(1,0) \in \overbrace{bd(D)}^{\text{boundary of } D}$ and $(1,0) \in D$.

$(0,1) \in bd(D)$ but $(0,1) \notin D$

$(0,0) \notin bd(D)$ but $(0,0) \in D$

$\boxed{\text{p. 928, def'n of bounded set}}$ ★ unrelated to boundary points!

e.g. $\{|x|+|y| \le 5\}$ is bounded, $\{y \ge 0\}$ is unbounded



$\boxed{\text{p. 928, Thm 8}}$ A cts. function on a $\overbrace{\text{closed, bounded set}}^{\text{called a "compact" set}}$ in $\mathbb{R}^n$ "attains its extrema".

$\boxed{\text{p. 929, Method 9}}$ Check boundary and $\overbrace{\text{interior}}^{\text{"non-boundary" points in } D}$ points separately!

(III) $\boxed{\text{p. 929, Exa. 7}}$

☆ It is sometimes easy to express the boundary of a region (or a part of it) parametrically, and then rewrite in terms of the parameters

e.g. the boundary of the disc $\{x^2+y^2 \leq 1\}$ is the circle $\{x^2+y^2=1\}$, which can be parametrized by $\mathbf{r}(\theta) = (\cos\theta, \sin\theta)$.

☆ In other cases, it is easier to work with implicit equations for the boundary. Then "Lagrange multiplers" (§14.8) can be used to find the extrema on the boundary.