Notes on directional derivatives, Stewart {14.6}

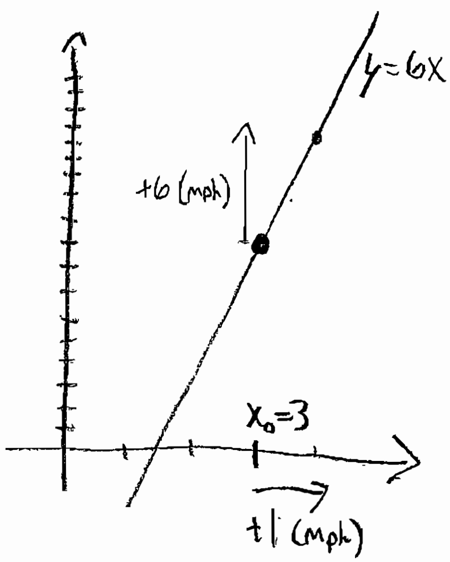
[Andrew Critch, Math 53, 2009 Summer]

In single-variable calculus,

sneaky coefficient of +1 for later!

$$f'(x_0) := \lim_{h\to 0} \frac{f(x_0 + 1h) - f(x_0)}{h}$$

Example 1 | $f(x) = 6x - 9$, $x_0 = 3$, $f'(3) = 6$. Here are two ways to conceptualize this situation:

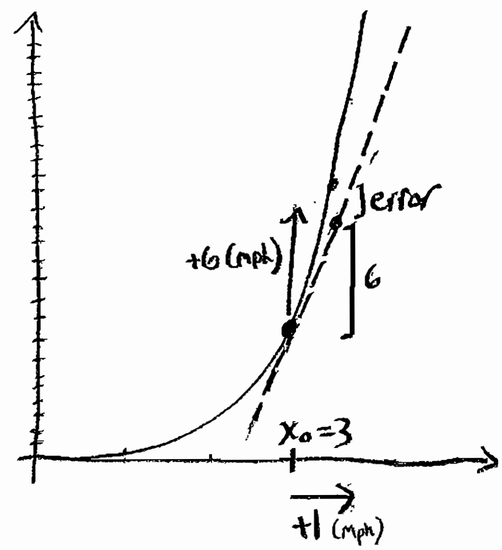


a) If we change $x_0 = 3$ by a total of +1, f changes by a total of +6.

b) If we change $x_0 = 3$ at a velocity of +1 (mph) (for an instant), f changes at at a velocity of +6 (mph) (for that instant).

Option (b) is a more accurate interpretation in non-linear examples (see below). Let us write $(D_{+1} f)(x_0)$ instead of $f'(x_0)$, for now to emphasize the sneaky h-coefficient of +1. This is called "the directional derivative of f along the vector +1 (at x_0)"

Example 2 $f(x)=x^2$, $x_0=3$, $f'(3)=6$.



a) If we change $x_0=3$ by a total of +1 f changes by approximately +6, with an error since the graph is curved. Hence "total changes" are not quite accurate for conceptualizing derivatives.

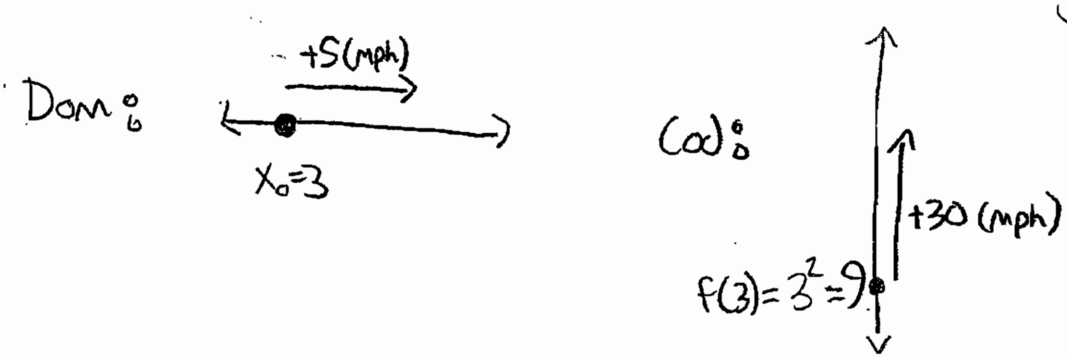
b) If we change $x_0=3$ at a velocity of +1 (mph) (for an instant!), f changes at a velocity of exactly +6 (for that instant).

This makes sense if we replace +1 by, say, +5:
If we change $x_0=3$ at a velocity of $+5=5(+1)$, then f changes at a velocity of $+30=5(+6)$:

$$(D_{+5}f)(3) := \lim_{h\to 0} \frac{f(x_0+5h)-f(x_0)}{h} = +30$$

(Check this!)

It is possible to visualize this situation entirely in terms of motion in the domain and codomain, without the graph:



(Not having to visualize a graph is nice when its in 4-D!)

These concepts all make sense if $f: \mathbb{R}^n \to \mathbb{R}^1$ and $+5$ is replaced by an arbitrary vector $\vec{v} \in \mathbb{R}^n$! We define:

$$(D_{\vec{v}} f)(\vec{x}_0) := \lim_{h \to 0} \frac{f(\vec{x}_0 + \vec{v} \cdot h) - f(\vec{x}_0)}{h}$$

" the <u>directional derivative</u> of f along the vector $\vec{v}$ (at $\vec{x}_0$) "

(☆ Note: Stewart requires that $|\vec{v}| = 1$, but I do not. Non-unit directional derivatives are everywhere in differential geometry and physics!)

The interpretation is the same as in (b) above:

$$\overbrace{(D_{\vec{v}} f)}^{\text{velocity of output change}}(\vec{x}_0)$$

Velocity of input change

input

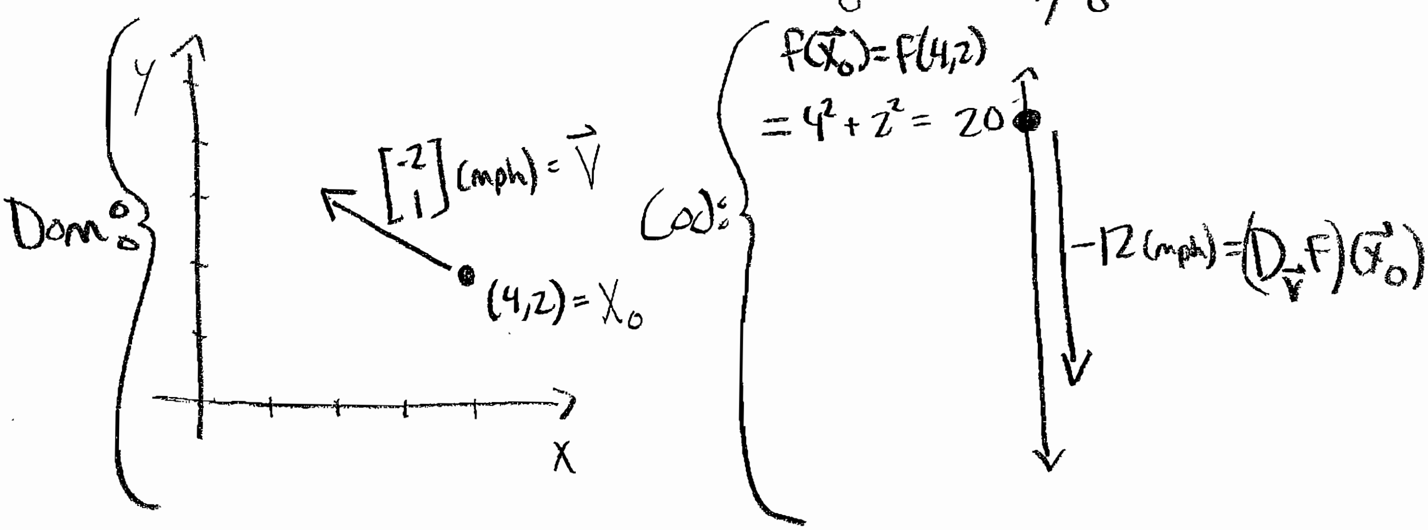
<u>Example 3</u> | $f(x,y)=x^2+y^2$, $\vec{x}_0=(4,2)$, and $\vec{v}=\begin{bmatrix}-2\\1\end{bmatrix}$:

$$\left(D_{\begin{bmatrix}-2\\1\end{bmatrix}} f\right)(4,2) := \lim_{h\to 0} \frac{f\left(\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}-2\\1\end{bmatrix}h\right)-f\left(\begin{bmatrix}4\\2\end{bmatrix}\right)}{h}$$

$$= \lim_{''} \frac{\left[(4-2h)^2+(2+h)^2\right]-\left[4^2+2^2\right]}{h}$$

$$= \lim_{''} \frac{-16h+4h^2+4h+h^2}{h} = \boxed{-12}$$

Here is the corresponding picture of the input, input change velocity and output change velocity:



When $\vec{v}$ is a unit vector, Stewart 14.3 gives a nice description of $(D_{\vec{v}} f)(\vec{x}_0)$ in terms of graph slopes, which I also recommend reading. But don't forget, slope is not the only way to think of derivatives!

One advantage of allowing non-unit directional derivatives is that we can write formulas such as

$$\left.\begin{aligned} D_{\vec{u}+\vec{v}} f &= D_{\vec{u}} f + D_{\vec{v}} f \\ \text{and} \quad D_{c\vec{v}} f &= c D_{\vec{v}} f \end{aligned}\right\} \text{"linearity" of directional derivatives}$$

which are in fact <u>true</u>! These identities are very important in physics and differential geometry. They will also help to understand the relationship between the "gradient" and directional derivatives.

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Notes on partial derivatives, Stewart 14.3

[Andrew Critch, Math 53, 2009 Summer]

Partial derivatives are a special case of directional derivatives! Just as in $\mathbb{R}^1$ our favorite direction is $+1$, in $\mathbb{R}^2$ our favorites are $\begin{bmatrix}1\\0\end{bmatrix}=\vec{i}$ and $\begin{bmatrix}0\\1\end{bmatrix}=\vec{j}$, and in $\mathbb{R}^3$ we like $\begin{bmatrix}1\\0\\0\end{bmatrix}=\vec{i}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}=\vec{j}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}=\vec{k}$, etc.

We define the <u>partial derivative</u> of f "in the x direction" or "with respect to x" by

$$\frac{\partial}{\partial x}f = D_{\vec{i}}f \text{ , and similarly } \frac{\partial}{\partial y}f = D_{\vec{j}}f\text{, etc.}$$

So, if $f:\mathbb{R}^2\to\mathbb{R}^1$, we have

$$\left(\frac{\partial}{\partial x}f\right)(x_0,y_0) = \lim_{h\to 0}\frac{f\left(\begin{bmatrix}x_0\\y_0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}h\right)-f\left(\begin{bmatrix}x_0\\y_0\end{bmatrix}\right)}{h}$$

$$= \boxed{\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}}$$

(direct definition)

This is just a single-variable derivative with a ",y_0" in it!

Thus, $\frac{\partial}{\partial x} f$ is computed by "holding y constant." After all, changing the input in the $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ direction does not change its y-value!

Examples | $\frac{\partial}{\partial x}\left(3x^2 \underbrace{y}_{\text{hold } y \text{ constant}}\right) = 6xy$ } compare $\frac{\partial}{\partial x}(3x^2 \cdot 5) = 6x \cdot 5$

$\frac{\partial}{\partial y}(e^{xy}) = xe^{xy}$ } compare $\frac{\partial}{\partial y}(e^{2y}) = 2e^{2y}$

etc.

Notation | For brevity (and sanity!), we write f_x in place of $\frac{\partial}{\partial x}$, and $(f_x)_y = $ "f_{xy}" in place of $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right)$. Note in the shorthand, the diffentions are written "from left to right." Other notations:

$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2 f}{\partial y \partial x}, \text{ etc.}$$

In "nice" situations, the order of diffentiation in fact does not matter!

Clairaut's Theorem If f_{xy} and f_{yx} (exist and) are continuous at a given input, then $\boxed{f_{xy} = f_{yx}}$ at that input.

Similarly, if for example f_{xxyz} and f_{zxyx} (exist and) are continuous at an input, they are equal at that input.

Note: There is nothing special about the x, y and z directions here! Derivatives such as $D_{\vec{u}}(D_{\vec{v}}f)$ and $D_{\vec{v}}(D_{\vec{u}}f)$ will agree in the same way.

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